A Formally Verified Abstract Account of Gödel's Incompleteness Theorems

Andrei Popescu

Middlesex University London

Dmitriy Traytel

ETH zürich

Gödel’s Incompleteness Theorems 1931

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Fix a consistent logical theory that

- contains enough arithmetic,
- can itself be arithmetized.

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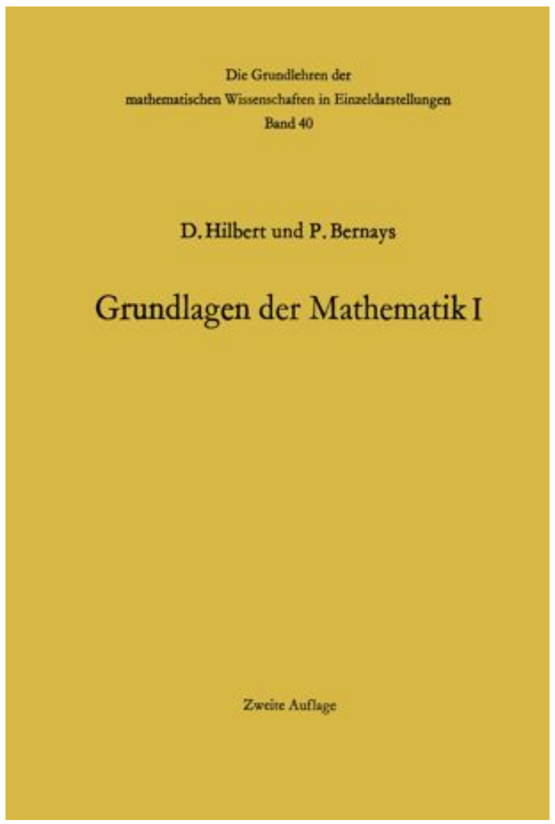
There are sentences that the theory cannot decide (i.e., neither prove nor disprove).

The theory cannot prove (an internal formulation of) its own consistency.

Pen and Paper Proofs of  and 

Pen and Paper Proofs of 1 and 2



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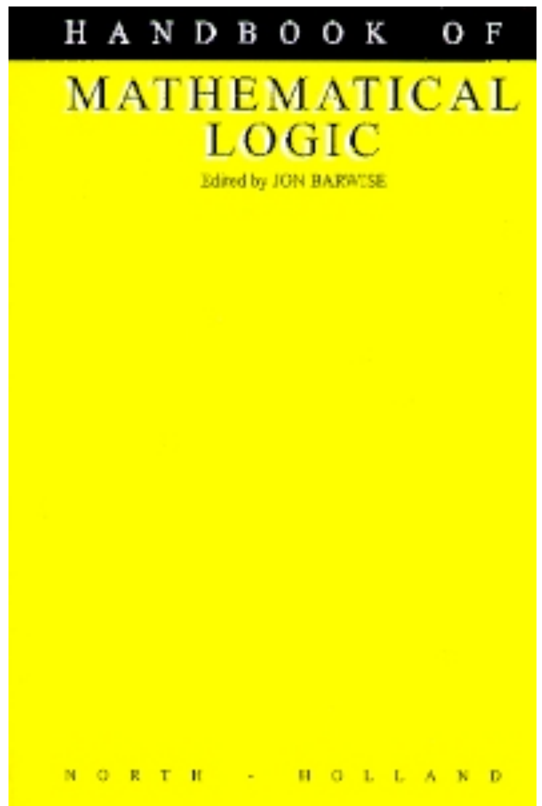

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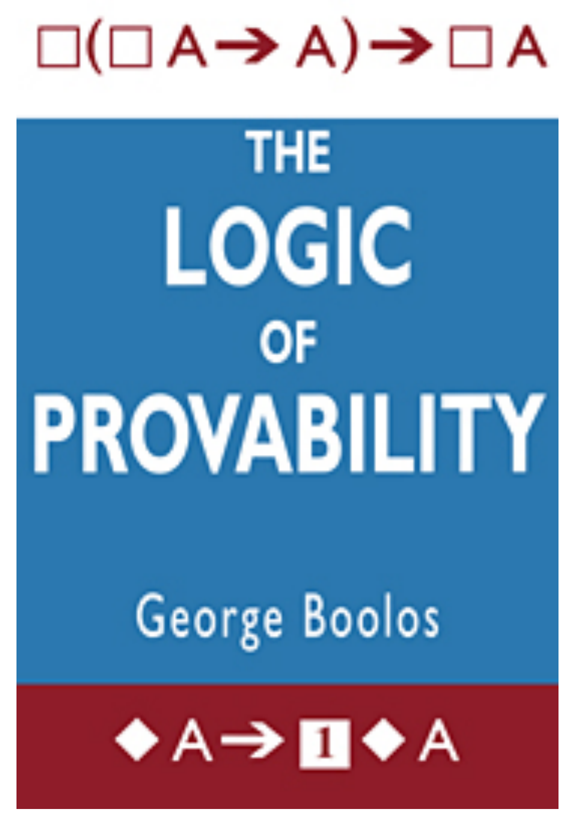

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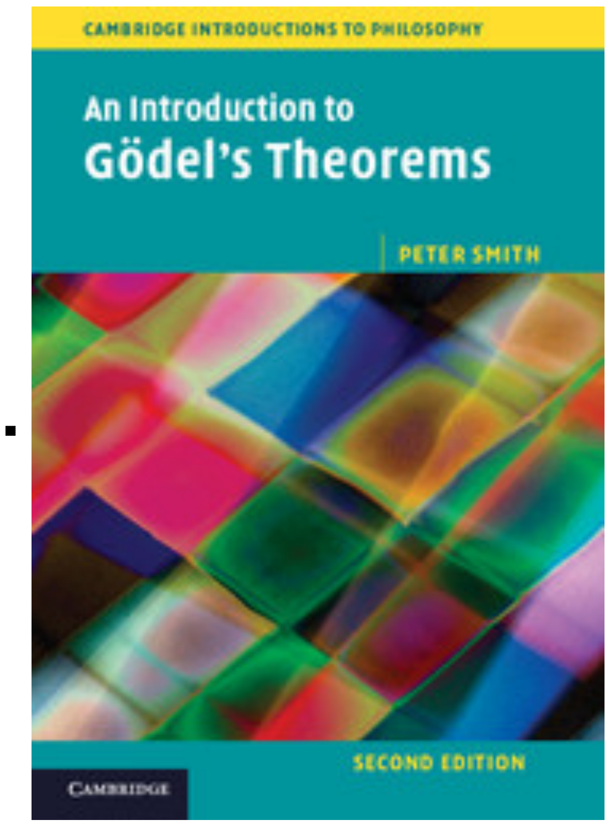

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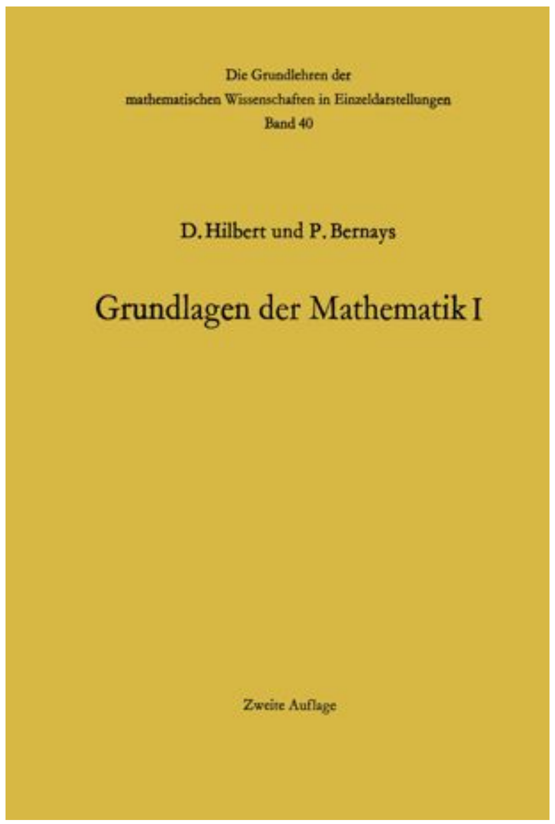

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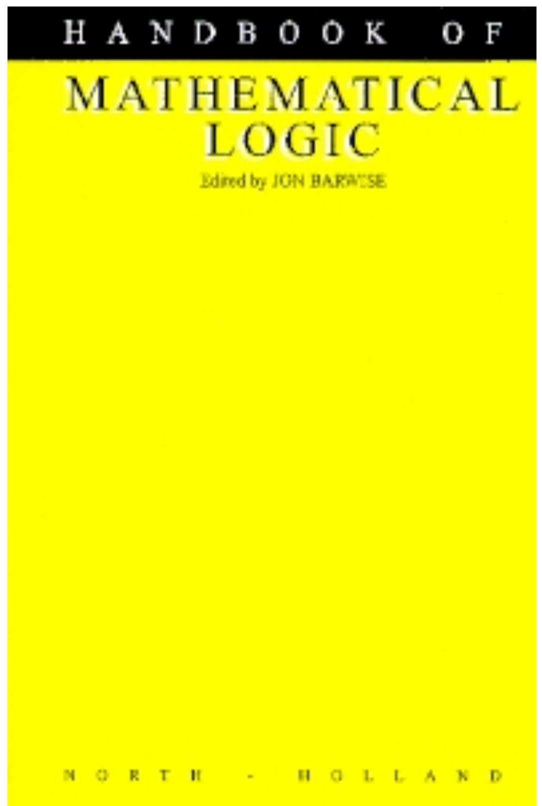

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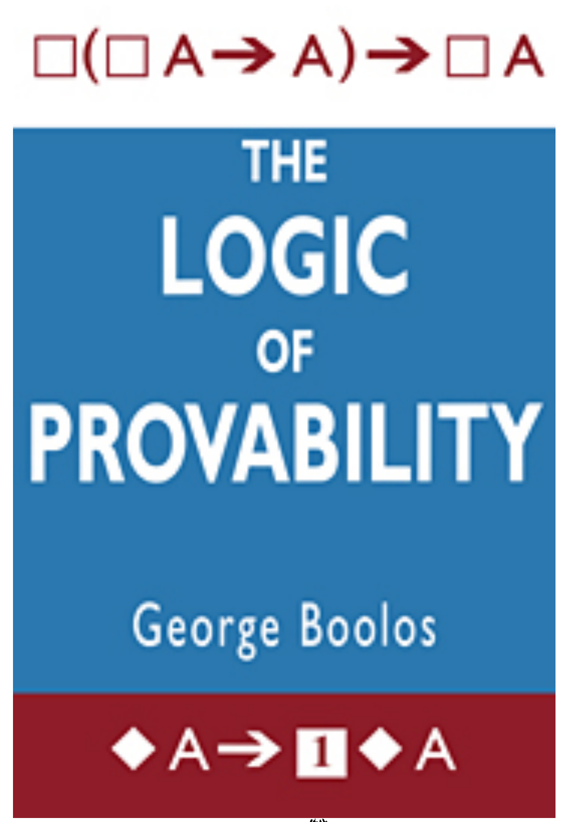

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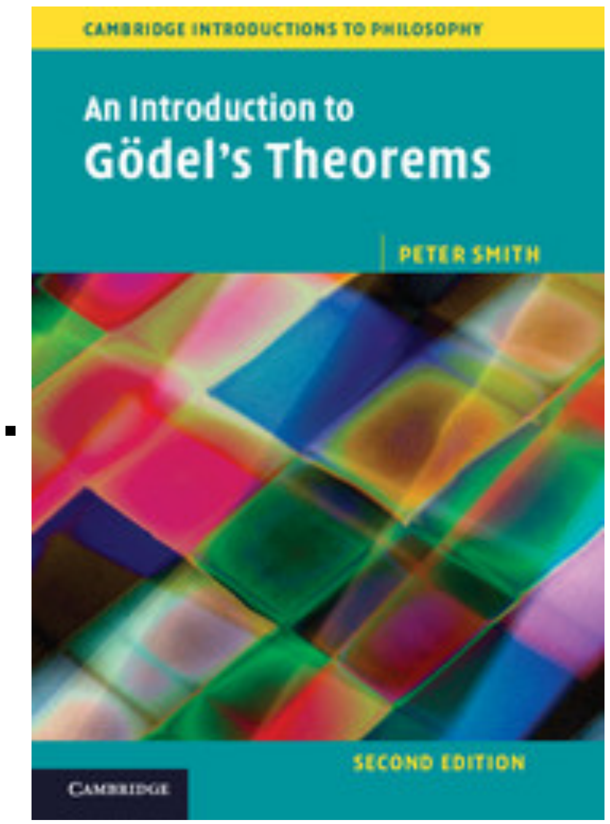
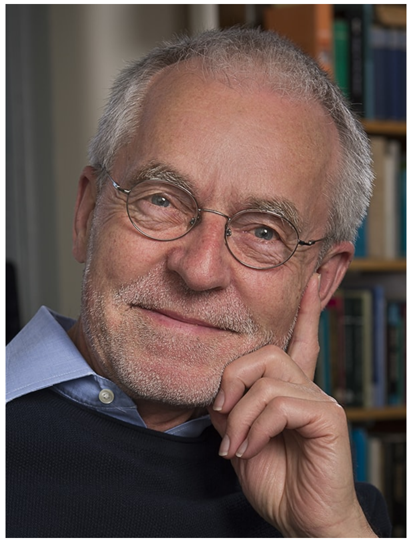


…

The reader who does not like **incomplete and (apparently) irremediably messy proofs** of syntactic facts may wish to skim over the rest of this chapter and take it for granted that …

Formal Verifications of  and 

Formal Verifications of 1 and 2

TEM	NQTHM	HOL Light	Coq	Isabelle
Sieg	Shankar	Harrison	O’Connor	Paulson
				
1978	1986	2004	2005	2015

End of story

End of story?

~~End of story?~~

Formal Verifications of 1 and 2

Shared structure

Formal Verifications of  and 

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- Fix a particular logic: **Classical FOL**

Formal Verifications of  and 

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- Fix a particular logic: **Classical FOL**
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Scope of 1 and 2 remains largely unexploded

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Scope of 1 and 2 remains largely unexploded

E.g. do they hold for **Intuitionistic FOL**, **HOL**, **CIC**?

Our Motto:

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Don’t Fix, Gather!

Our Contributions

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- Abstract formalization of  and 
  - **Answer** “What must/may a logic/theory offer?”
  - **Understand** variants and **distill** trade-offs from the literature
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  - **Reproduce** (for 1) and **improve** (for 2) Paulson’s formalization

What must a logic/theory offer?

Generic Syntax	Connectives	Provability Relation	Numerals

What must a logic/theory offer?

Generic Syntax	Connectives	Provability Relation	Numerals

What may a logic/theory offer?

Classical Logic	Order-like Relation	Proofs	Encodings
Represent-ability	Derivability Conditions	Standard Model	Soundness
Consistency	Omega-Consistency	Completeness of Provability	Proofs vs. Provability

Generic
Syntax

- **sets:** Var, Term, Fmla with Var⊆Term

Generic Syntax

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- **operators:**

FV_Term : Term → 2^{Var}

FV : Fmla → 2^{Var}

subst_Term : Term → Var → Term → Term

subst : Fmla → Var → Term → Fmla

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- **properties, e.g.:**

  $x \in FV(\phi)$ implies $FV(\text{subst}\ \phi\ x\ s) = FV(\phi) - \{x\} \cup FV\_Term(s)$

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We require unary substitution only.
We derive parallel substitution from it.

Connectives

- **operators:**

  $\equiv$: Term → Term → Fmla

  $\to$, $\wedge$, $\vee$: Fmla → Fmla → Fmla

  $\neg$: Fmla → Fmla

  $\bot$, $\top$: Fmla

  $\exists$, $\forall$: Var → Fmla → Fmla

Connectives

Connectives

- **operators:**

  $\equiv$: Term → Term → Fmla

  →, ∧, ∨ : Fmla → Fmla → Fmla

  ¬ : Fmla → Fmla

  ⊥, ⊤ : Fmla

  ∃, ∀ : Var → Fmla → Fmla

We require a minimal list w.r.t. intuitionistic deduction and define the rest.
Note: operators, not constructors

Provability Relation

- **unary relation:**

  $\vdash \subseteq \mathrm{Fmla}$

  we write $\vdash\phi$ if $\phi \in \vdash$

- **properties:**

  $\vdash$ contains the standard (Hilbert-style) intuitionistic FOL axioms about the connectives

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- **nonempty set:**

  $\text{Num} \subseteq \text{Fmla}_0$

Provability Relation

Numerals

- **property:** $\vdash \neg\neg\phi \to \phi$

Classical Logic

- **property:** $\vdash \neg\neg\phi \rightarrow \phi$

- **formula:** $< \in Fmla_2$
- **properties, e.g.:**

for all $\phi \in Fmla_1$ and $n \in Num$,

if $\vdash \phi(m)$ for all $m \in Num$, then $\vdash \forall x.\ x<n \rightarrow \phi(x)$

Classical Logic

Order-like Relation

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Order-like Relation

- **set:** Proof
- **binary relation:** $\Vdash\ \in Proof \times Fmla$

we write $p \Vdash \phi$ if $(p,\phi) \in \Vdash$

Proofs

- **operators:**

  $\langle \_ \rangle$: Fmla → Num and $\langle \_ \rangle$: Proof → Num

- **formulas** <u>subst</u>, <u>⊩</u>, <u>¬</u>
- **property:**

  behave like operators/relations (subst, ⊩, ¬) on encodings

Encodings

Represent-
ability

- **operators:**

  $\langle \_ \rangle : \mathrm{Fmla} \to \mathrm{Num}$ and $\langle \_ \rangle : \mathrm{Proof} \to \mathrm{Num}$

Encodings

- **formulas** $\underline{\text{subst}}$, $\underline{\Vdash}$, $\underline{\neg}$
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Represent-
ability

- **property:** $\nVdash \bot$

Consistency

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Represent-ability

- **property:** $\nvdash \perp$

Consistency

- **property:** For all $\phi \in \mathrm{Fmla}_1$,

  if $\vdash \neg\phi(n)$ for all $n \in \mathrm{Num}$ then $\nvdash \neg\neg(\exists x.\phi(x))$

Omega-Consistency

What must a logic/theory offer?

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Omega-Consistency

Proofs

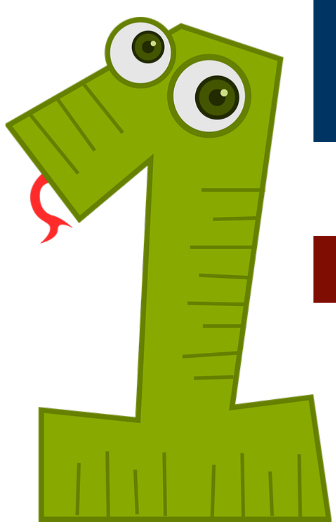
<u>subst</u>, $\underline{\Vdash}$

$\vdash \phi$ implies $\vdash \underline{\vdash} \langle \phi \rangle$

Represent-ability

Derivability Conditions

Encodings



There exists $\phi \in \mathrm{Fmla}_0$ such that $\nvdash \phi$ and $\nvdash \neg \phi$

Rosser's Trick

Consistency

Proofs

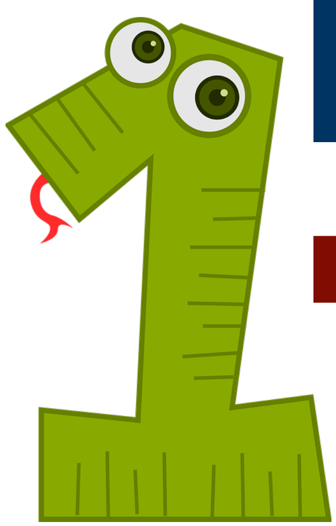
<u>subst, $\Vdash$</u>

Represent-ability

$\vdash \phi$ implies $\vdash \underline{\vdash} \langle \phi \rangle$

Derivability Conditions

Encodings



a la Rosser

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Rosser's Trick

Consistency

Proofs

$\underline{\text{subst}}$, $\underline{\Vdash}$

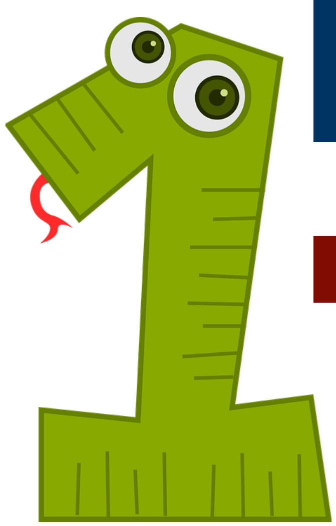
$\vdash\phi$ implies $\vdash\underline{\vdash}\langle\phi\rangle$

Represent-ability

$\underline{\neg}$

Derivability Conditions

Encodings



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Derivability Conditions

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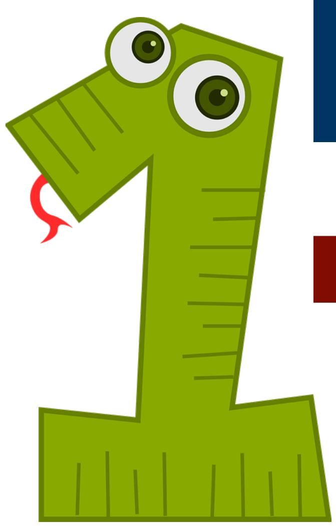

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Standard Model

Soundness

Completeness of Provability

Proofs vs. Provability

<u>subst</u>

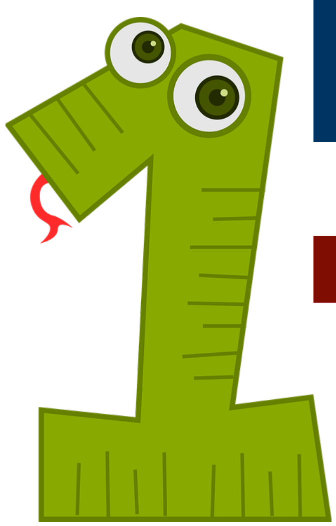
Represent-ability

$\vdash \phi$ implies $\vdash \underline{\vdash}\langle \phi \rangle$

Derivability Conditions

$\vdash \underline{\vdash}\langle \phi \rangle$ implies $\vdash \phi$

Encodings



semantic

There exists $\phi \in \mathrm{Fmla}_0$ such that

$\nvdash \phi$ and $\nvdash \neg\phi$

and ϕ is true in the standard model

Consistency

Classical Logic

<u>subst</u>

Representability

$\vdash \phi$ implies $\vdash \underline{\vdash}\langle \phi \rangle$

Derivability Conditions

$\vdash \underline{\vdash}\langle \phi \rangle$ implies $\vdash \phi$

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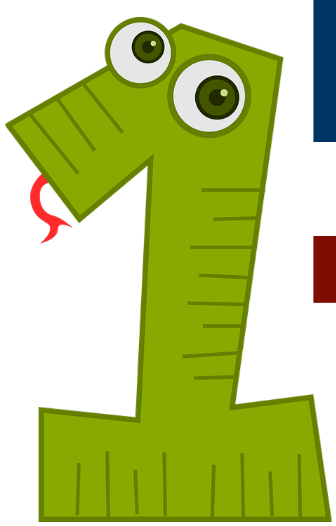

classical

There exists $\phi \in \mathrm{Fmla}_0$ such that $\nvdash \phi$ and $\nvdash \neg \phi$

Consistency

$\vdash \underline{\vdash}\langle\phi\rangle \to \underline{\vdash}\langle\underline{\vdash}\langle\phi\rangle\rangle$

$\vdash \underline{\vdash}\langle\phi\rangle \wedge \underline{\vdash}\langle\phi \to \psi\rangle \to \underline{\vdash}\langle\psi\rangle$

$\vdash\phi$ implies $\vdash \underline{\vdash}\langle\phi\rangle$

<u>subst</u>

Represent-
ability

Derivability
Conditions

Encodings

$$\nvdash \neg\underline{\vdash}\langle\bot\rangle$$

In the paper:
Jeroslow's
“improvement"
to remove this
condition

results in weaker
conclusion
\+ mistake in proof

Consistency

$\vdash \underline{\vdash}\langle\phi\rangle \to \underline{\vdash}\langle\underline{\vdash}\langle\phi\rangle\rangle$

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Represent-
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$$\nvdash \neg \underline{\vdash}\langle\bot\rangle$$

Summary Using our generic infrastructure (Section 2), we have formally proved several abstract incompleteness results. They include four versions of $\mathcal{IT}_1$:

- Gödel's original $\mathcal{IT}_1$ (Theorem 9) and an $\mathcal{IT}_1$ based on classical logic (Theorem 12) required the formalization of some well-known arguments without change.
- Rosser's $\mathcal{IT}_1$ (Theorem 10) involved the generalization of a well-known argument: distilling two abstract conditions, Ord_1 and Ord_2.
- Novel semantic variants of $\mathcal{IT}_1$ (Theorems 11 and 13) were born from abstractly connecting standard models, HBL_1's "iff" version, and proof representability.

They also include two versions of $\mathcal{IT}_2$:

- The standard $\mathcal{IT}_2$ based on the three derivability conditions (Theorem 14) again only required formalizing a well-known argument.
- The alternative, Jeroslow-style $\mathcal{IT}_2$ (Theorems 17 and 18) involved a detailed analysis and correction of an existing abstract result.

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12000 LOC

From Abstract to Concrete

Generic Syntax	Connectives	Provability Relation	Numerals

Verified instances

- Robinson’s Arithmetic (Q)
- Hereditarily finite set theory

From Abstract to Concrete

Instantiations of

classical
semantic

with Paulson's
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12000 LOC

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with Paulson's HF set theory.

Paulson assumes soundness (and redundantly consistency!)

We removed the soundness assumption from the instantiation of 

→ strictly stronger result

→ required us to replace "easy" semantic proofs with tedious proofs in the HF calculus (no help from the abstract side here)

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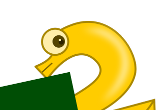

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+5000 LOC

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Conclusion

- Abstract formalization of 1 and 2
  - **Answer** "What must/may a logic/theory offer?"
  - **Understand** variants and **distill** trade-offs from the literature
  - **Correct** a mistake in a pen and paper proof
- Concrete instantiation to hereditarily finite set theory
  - **Reproduce** (for 1) and **improve** (for 2) Paulson's formalization
- Still unanswered/future work
  - Do 1 and 2 hold for Intuitionistic FOL, HOL, CIC?
  - Can we do more on the abstract level? (e.g. derivability conditions)

Thank you!
Questions?

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