Verified Decision Procedures

for

Monadic Second-Order Logic on Strings

Functional Pearl

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Overview

MSO

Overview

MSO

$\mathscr{L}_{\mathrm{MSO}}(\varphi) = \mathscr{L}_{\mathrm{MSO}}(\psi)$?

Overview

Finite Automata

MSO

$\mathscr{L}_{\mathsf{MSO}}(\varphi) = \mathscr{L}_{\mathsf{MSO}}(\psi)$?

Overview

$$\begin{array}{c}\boxed{\text{Finite Automata}} \\ \Big\uparrow \text{ MONA } (> 40 \text{ kLOC of C/C++}) \\ \boxed{\text{MSO}}\end{array}$$

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Finite Automata

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Regular Expressions

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Finite Automata

$$\begin{tikzcd} \text{Finite Automata} \\ \text{MSO} \arrow[u, "\text{MONA } (> 40 \text{ kLOC of C/C++})"'] \end{tikzcd}$$

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Regular Expressions

$\mathscr{L}(\alpha) = \mathscr{L}(\beta)$?

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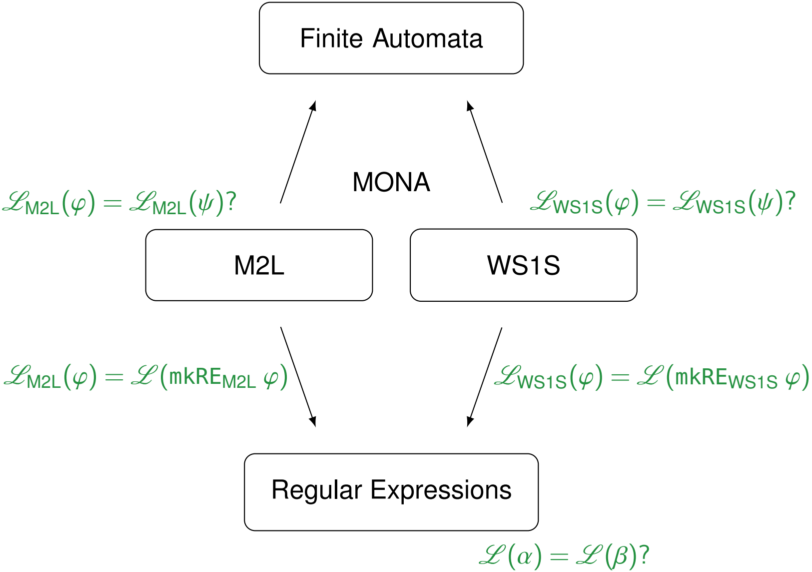
$\mathscr{L}_{\mathsf{MSO}}(\varphi) = \mathscr{L}_{\mathsf{MSO}}(\psi)$?

$\mathscr{L}_{\mathsf{MSO}}(\varphi) = \mathscr{L}(\texttt{mkRE}\ \varphi)$

Regular Expressions

$\mathscr{L}(\alpha) = \mathscr{L}(\beta)$?

Overview


Finite Automata
MONA
$\mathscr{L}_{\mathsf{M2L}}(\varphi) = \mathscr{L}_{\mathsf{M2L}}(\psi)$?
$\mathscr{L}_{\mathsf{WS1S}}(\varphi) = \mathscr{L}_{\mathsf{WS1S}}(\psi)$?
M2L
WS1S
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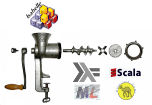
M2L

WS1S



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$\mathscr{L}(\alpha) = \mathscr{L}(\beta)$?

Outline

Regular Expressions Equivalence

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Regular Expressions

$$
\begin{aligned}
\mathscr{L}(\varnothing) &= \{\} \\
\mathscr{L}(\varepsilon) &= \{[\,]\} \\
\mathscr{L}(a) &= \{[a]\} && a \in \Sigma \\
\mathscr{L}(\alpha + \beta) &= \mathscr{L}(\alpha) \cup \mathscr{L}(\beta) \\
\mathscr{L}(\alpha \cdot \beta) &= \mathscr{L}(\alpha) \cdot \mathscr{L}(\beta) \\
\mathscr{L}(\alpha^*) &= \mathscr{L}(\alpha)^*
\end{aligned}
$$

Extended Regular Expressions

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\mathscr{L}(\alpha \cap \beta) &= \mathscr{L}(\alpha) \cap \mathscr{L}(\beta) \\
\mathscr{L}(\neg\, \alpha) &= \Sigma^* \setminus \mathscr{L}(\alpha)
\end{aligned}
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Π-Extended Regular Expressions

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\mathscr{L}(\Pi\,\alpha) &= \{ \qquad w \mid w \in \mathscr{L} \qquad (\alpha)\}
\end{aligned}
$$

Π-Extended Regular Expressions

$$
\begin{array}{rcll}
\mathscr{L}_n(\varnothing) & = & \{\} & \\
\mathscr{L}_n(\varepsilon) & = & \{[]\} & \\
\mathscr{L}_n(a) & = & \{[a]\} & a \in \Sigma_n \\
\mathscr{L}_n(\alpha + \beta) & = & \mathscr{L}_n(\alpha) \cup \mathscr{L}_n(\beta) & \\
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\mathscr{L}_n(\emptyset) &= \{\} \\
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Example $\Sigma_n = \{\top, \bot\}^n$ $\left[\begin{array}{c|c|c} \top & \bot & \bot \\ \bot & \top & \top \\ \bot & \bot & \top \end{array}\right] \in \Sigma_3^*$

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Example $\Sigma_n = \{\top, \bot\}^n$ $\quad \left[\begin{array}{c|c|c} \cancel{\top} & \cancel{\bot} & \cancel{\bot} \\ \bot & \top & \top \\ \bot & \bot & \top \end{array}\right] \in \Sigma_2^*$

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\end{array}
$$

$$\pi : \Sigma_{n+1} \to \Sigma_n$$

Example $\Sigma_n = \{\top, \bot\}^n$ $\left[\begin{array}{c|c|c} \cancel{\top} & \cancel{\bot} & \cancel{\bot} \\ \bot & \top & \top \\ \bot & \bot & \top \end{array}\right] \in \Sigma_2^*$ $\quad \pi = \mathtt{tail}$, $\pi^{-1} a = \{\top a, \bot a\}$

Derivatives of Regular Expressions

Characteristic property $\mathscr{L}_n(\mathscr{D}_a(\alpha)) = \{w \mid aw \in \mathscr{L}_n(\alpha)\}$

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\mathscr{D}_a(\Pi\, \alpha) &= \Pi \left(\bigoplus_{b \in \pi^{-1} a} \mathscr{D}_b(\alpha) \right)
\end{aligned}
$$

DP by Example: $a^* \stackrel{?}{=} \varepsilon + a \cdot a^*$ for $\Sigma = \{a, b\}$

$$\boxed{\begin{array}{c} a^* \\ \hline \varepsilon + a \cdot a^* \end{array}}$$

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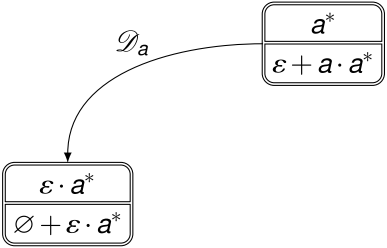

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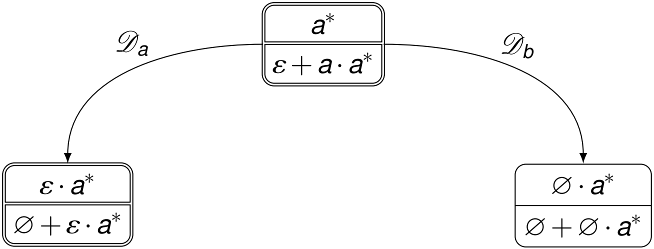

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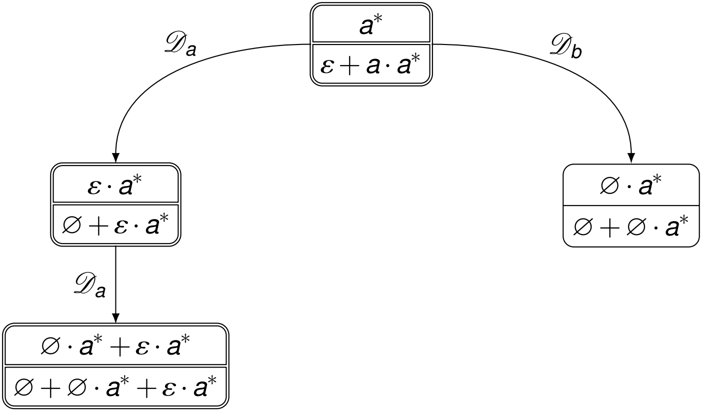

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$$
\begin{tikzcd}
 & \boxed{\begin{array}{c} a^* \\ \hline \varepsilon + a \cdot a^* \end{array}} \arrow[dl, bend right, "\mathscr{D}_a"'] \arrow[dr, bend left, "\mathscr{D}_b"] & \\
\boxed{\begin{array}{c} \varepsilon \cdot a^* \\ \hline \varnothing + \varepsilon \cdot a^* \end{array}} \arrow[d, "\mathscr{D}_a"'] & & \boxed{\begin{array}{c} \varnothing \cdot a^* \\ \hline \varnothing + \varnothing \cdot a^* \end{array}} \\
\boxed{\begin{array}{c} \varnothing \cdot a^* + \varepsilon \cdot a^* \\ \hline \varnothing + \varnothing \cdot a^* + \varepsilon \cdot a^* \end{array}} \arrow[d, bend right, "\mathscr{D}_a"'] & & \\
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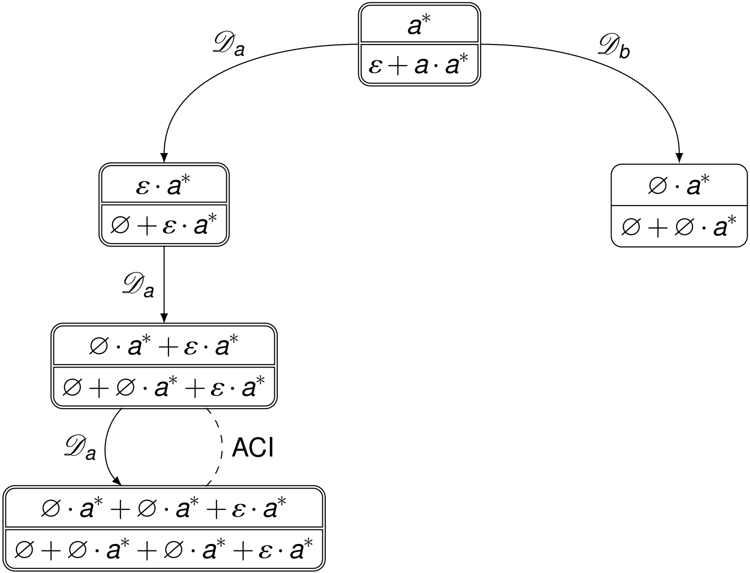

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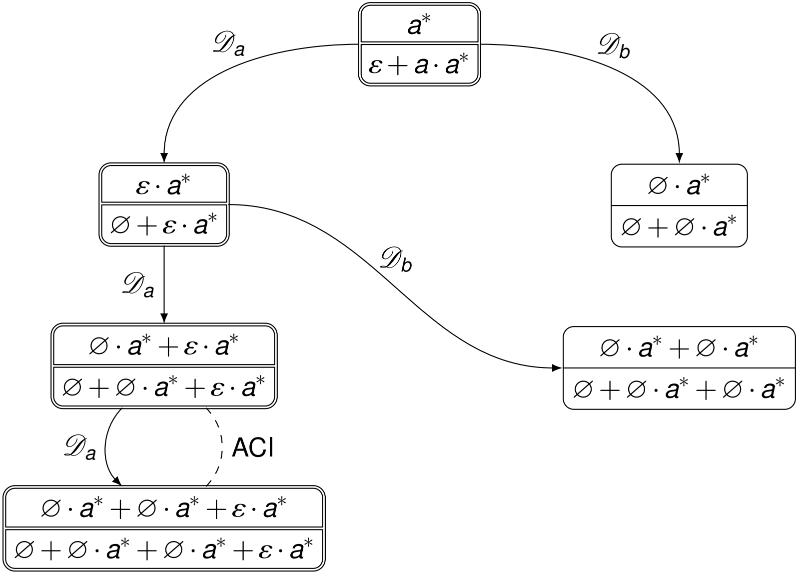

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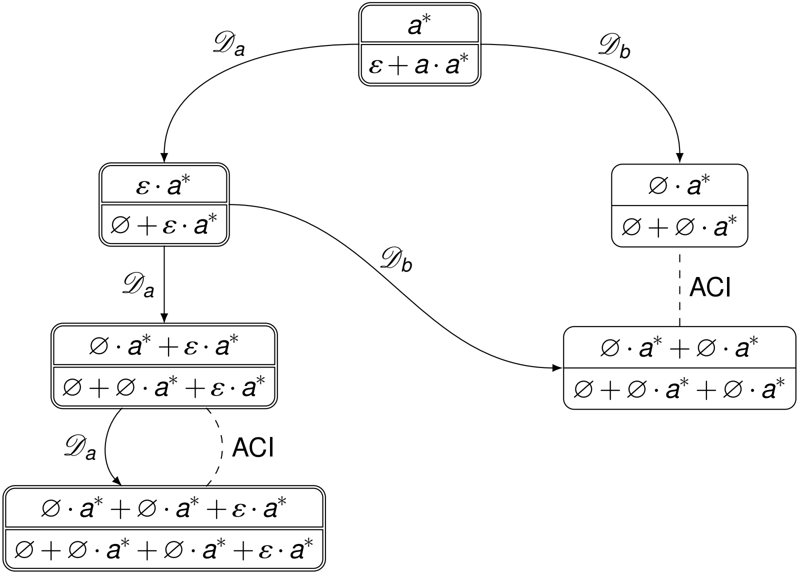

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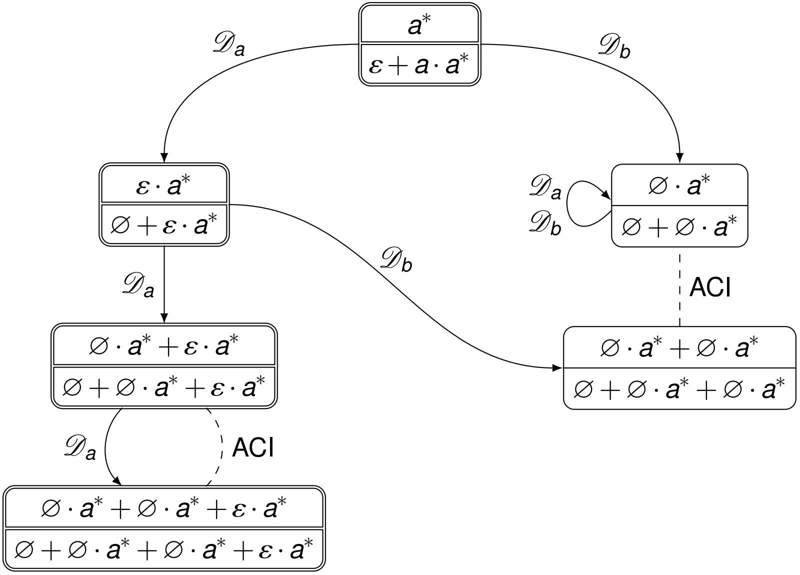

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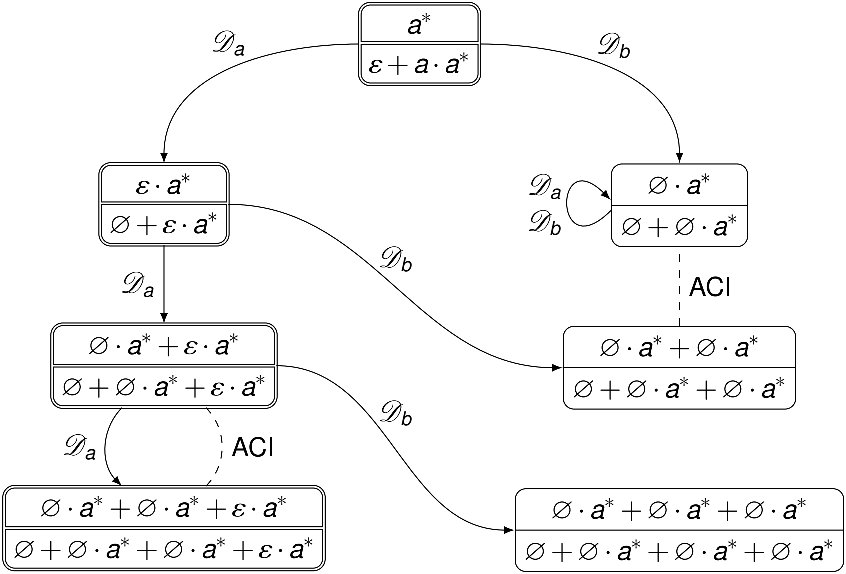

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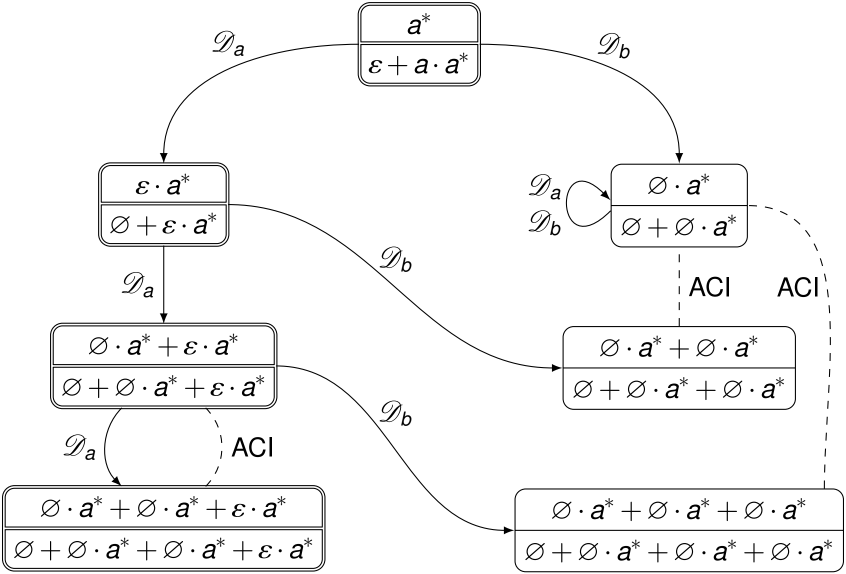

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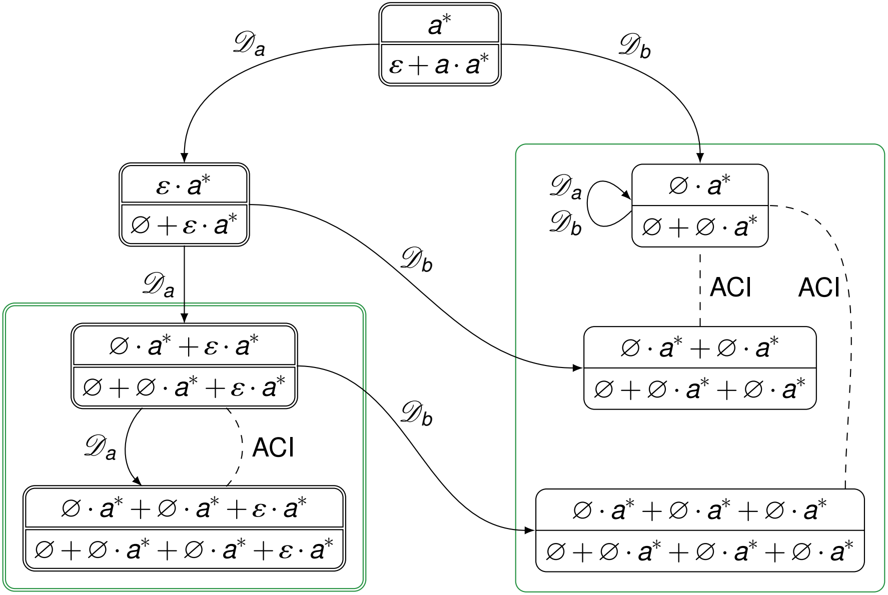

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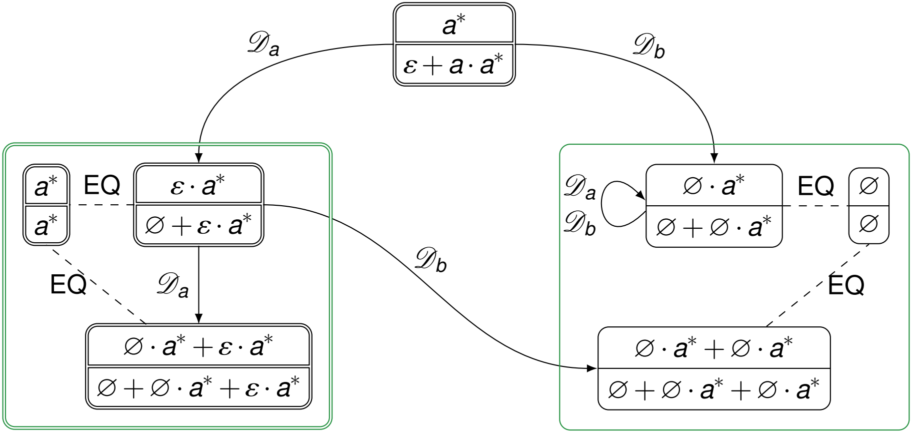

Related Work

- Theoretical groundwork

JACM 1964	Brzozowski
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- FP community

JFP 2009	Owens, Reppy, and Turon
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- ITP community

ITP 2010	Braibant and Pous
JAR 2011	Krauss and Nipkow
CPP 2011	Coquand and Siles
ITP 2012	Asperti
RAMiCS 2012	Moreira, Pereira, and de Sousa

Outline

Regular Expressions Equivalence

MSO

MSO Formulas

$$
\begin{array}{rcl}
\texttt{formula} & = & \mathrm{Q}_a(x) \\
& \mid & x < y \\
& \mid & x \in X \\
& \mid & \neg\, \texttt{formula} \\
& \mid & \texttt{formula} \vee \texttt{formula} \\
& \mid & \texttt{formula} \wedge \texttt{formula} \\
& \mid & \exists x.\texttt{formula} \\
& \mid & \exists X.\texttt{formula}
\end{array}
$$

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\end{array}
$$

$$(w, \mathfrak{I}) \models \mathsf{Q}_a(x) \;\Leftrightarrow\; w \;!!\; \mathfrak{I}\, x = a$$

MSO Formulas

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& | & \neg\ \texttt{formula} \\
& | & \texttt{formula} \lor \texttt{formula} \\
& | & \texttt{formula} \land \texttt{formula} \\
& | & \exists x.\,\texttt{formula} \\
& | & \exists X.\,\texttt{formula}
\end{array}
$$

$$(w, \mathfrak{I}) \vDash \mathrm{Q}_a(x) \;\Leftrightarrow\; w \,\texttt{!!}\, \mathfrak{I}\, x = a$$

$$\mathscr{L}_{\mathrm{M2L}}(\varphi) = \{\mathsf{enc}(w, \mathfrak{I}) \mid (w, \mathfrak{I}) \vDash \varphi\}$$

Representation of Interpretations as Words

$$\big(w = aba, \qquad \mathfrak{I} = \{x \mapsto 0,\ X \mapsto \{1,2\},\ y \mapsto 2\}\big)$$

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$$\Big\downarrow \mathsf{enc}$$

$$\begin{array}{cc} & \\ x & \\ X & \\ y & \end{array}\left[\begin{array}{c|c|c} a & b & a \\ \top & \bot & \bot \\ \bot & \top & \top \\ \bot & \bot & \top \end{array}\right]$$

$$\Sigma_n = \Sigma \times \{\top, \bot\}^n$$

Representation of Interpretations as Words

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$$\Sigma_n = \Sigma \times \{\top, \bot\}^n$$

$$\pi\,(a, bs) \;=\; (a, \texttt{tail}\, bs)$$

$$\pi^{-1}(a, bs) \;=\; \{(a, \top bs), (a, \bot bs)\}$$

From MSO Formulas to Regular Expressions

$$\mathtt{mkRE}\ n\,(\mathsf{Q}_a(m)) \;=\; \Sigma_n^* \cdot \begin{pmatrix} a \\ {}^{\top}\!/\!\bot \\ \top \\ {}^{\top}\!/\!\bot \end{pmatrix} \cdot \Sigma_n^* \cap \mathtt{WF}\ n\ \{m\}$$

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$$\vdots$$

$$\mathtt{mkRE}\; n\,(\varphi_1 \vee \varphi_2) \;=\; (\mathtt{mkRE}\; n\,\varphi_1 + \mathtt{mkRE}\; n\,\varphi_2) \cap \mathtt{WF}\; n\,(\mathtt{FV}\,(\varphi_1 \vee \varphi_2))$$

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\mathtt{mkRE}\ n\,(\mathrm{Q}_a(m)) &= \Sigma_n^* \cdot \begin{pmatrix} a \\ {}^\top\!/\!_\perp \\ \top \\ {}^\top\!/\!_\perp \end{pmatrix} \cdot \Sigma_n^* \cap \mathtt{WF}\ n\ \{m\} \\
&\ \vdots \\
\mathtt{mkRE}\ n\,(\varphi_1 \vee \varphi_2) &= (\mathtt{mkRE}\ n\,\varphi_1 + \mathtt{mkRE}\ n\,\varphi_2) \cap \mathtt{WF}\ n\,(\mathtt{FV}\ (\varphi_1 \vee \varphi_2)) \\
&\ \vdots \\
\mathtt{mkRE}\ n\,(\exists x.\varphi) &= \Pi\,(\mathtt{mkRE}\ (n+1)\ \varphi) \\
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\end{aligned}
$$

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\texttt{mkRE}\ n\,(\mathrm{Q}_a(m)) &= \Sigma_n^* \cdot \begin{pmatrix} a \\ {}^{\top}/{}_{\bot} \\ \top \\ {}^{\top}/{}_{\bot} \end{pmatrix} \cdot \Sigma_n^* \\
&\ \vdots \\
\texttt{mkRE}\ n\,(\varphi_1 \vee \varphi_2) &= \texttt{mkRE}\ n\,\varphi_1 + \texttt{mkRE}\ n\,\varphi_2 \\
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$$

From MSO Formulas to Regular Expressions

$$\mathtt{mkRE}\ n\ (\mathrm{Q}_a(m)) \ = \ \Sigma_n^* \cdot \begin{pmatrix} a \\ \top/\bot \\ \top \\ \top/\bot \end{pmatrix} \cdot \Sigma_n^*$$

$$\vdots$$

$$\mathtt{mkRE}\ n\ (\varphi_1 \vee \varphi_2) \ = \ \mathtt{mkRE}\ n\ \varphi_1 + \mathtt{mkRE}\ n\ \varphi_2$$

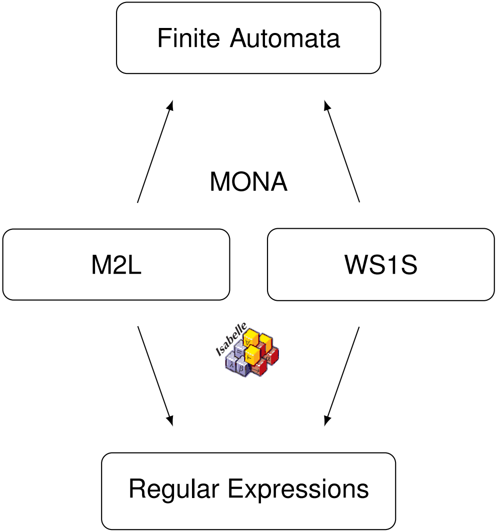
$$\vdots$$

$$\mathtt{mkRE}\ n\ (\exists x.\varphi) \ = \ \Pi\ (\mathtt{mkRE}\ (n+1)\ \varphi \cap \mathtt{WF}\ (n+1)\ \{x\})$$

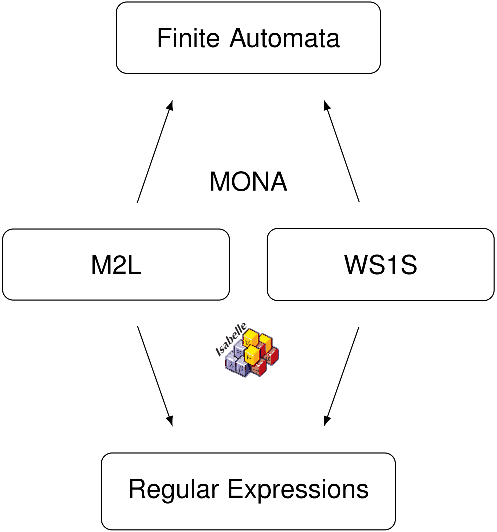
$$\mathtt{mkRE}\ n\ (\exists X.\varphi) \ = \ \Pi\ (\mathtt{mkRE}\ (n+1)\ \varphi)$$

Theorem $\mathscr{L}_{\mathrm{M2L}}(\varphi) = \mathscr{L}_n(\mathtt{mkRE}\ n\ \varphi \cap \mathtt{WF}\ n\ (\mathtt{FV}\ \varphi)) - \{\varepsilon\}$

Head to Head

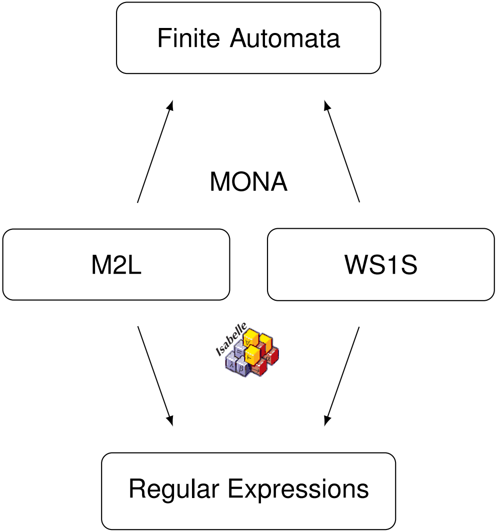

Finite Automata
MONA
M2L
WS1S
Isabelle
Regular Expressions

Head to Head



☺ efficient

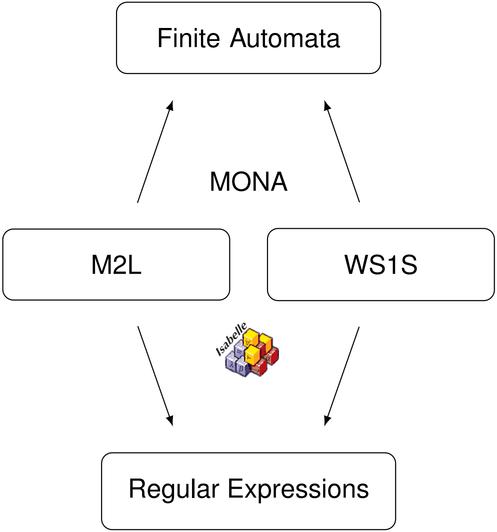
Head to Head



☺ efficient

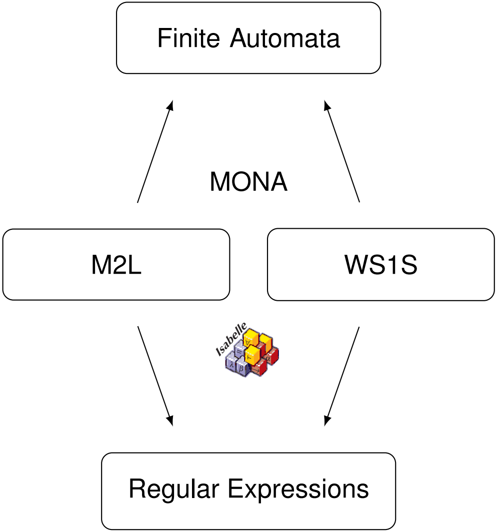
☹ unverified

Head to Head



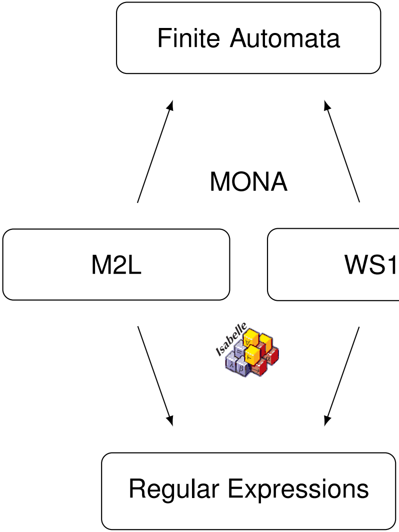
- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC

Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- ☹ 40 kLOC of C/C++

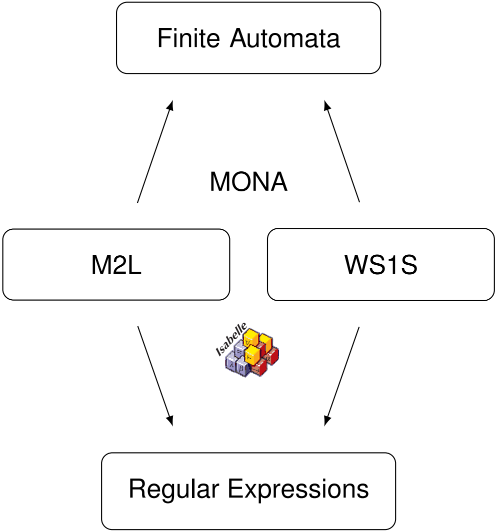
Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- ☹ 40 kLOC of C/C++

- ☹ inefficient

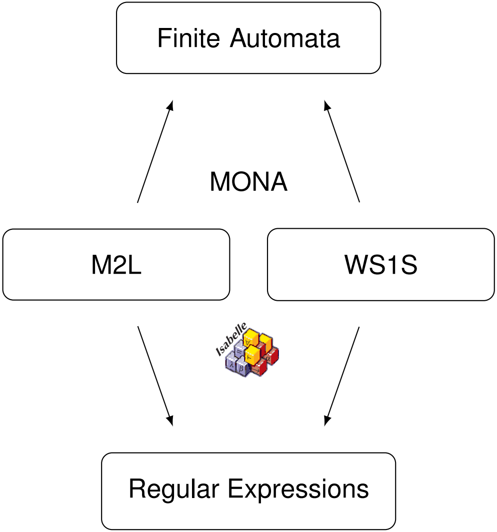
Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- ☹ 40 kLOC of C/C++

- ☹ inefficient
- ☺ simple: 350 LOC of Isabelle/HOL

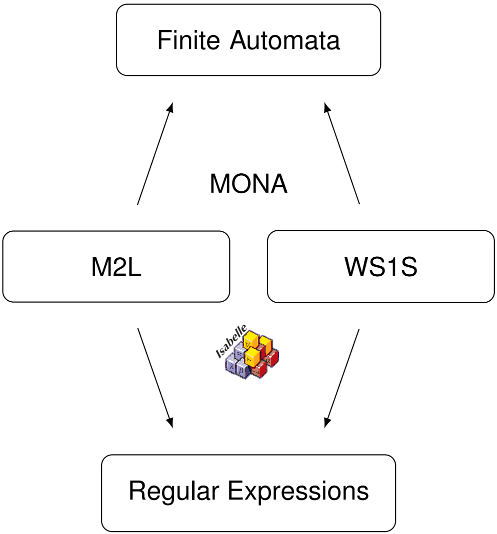
Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- ☹ 40 kLOC of C/C++

- ☹ inefficient
- ☺ simple: 350 LOC of Isabelle/HOL
- ☺ functional: 2,5 kLOC of generated Haskell

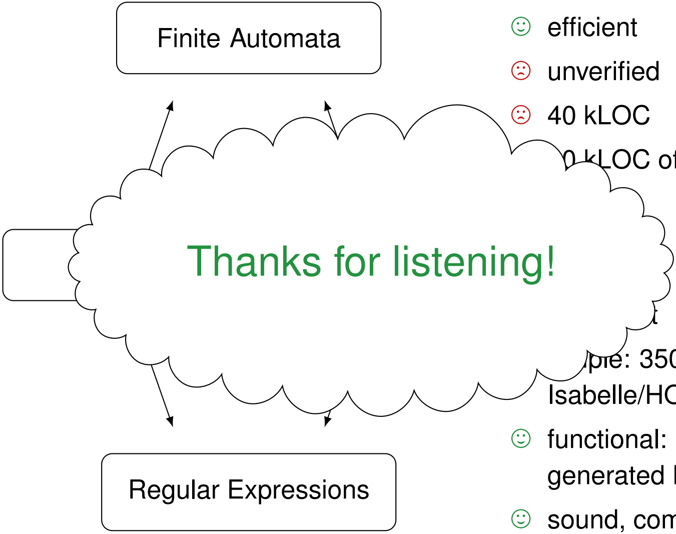
Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- ☹ 40 kLOC of C/C++

- ☹ inefficient
- ☺ simple: 350 LOC of Isabelle/HOL
- ☺ functional: 2,5 kLOC of generated Haskell
- ☺ sound, complete, and terminating: 5 kLOP

Head to Head



- ☺ efficient
- ☹ unverified
- ☹ 40 kLOC
- 0 kLOC of C/C++
- t
- ple: 350 LOC of Isabelle/HOL
- ☺ functional: 2, 5 kLOC of generated Haskell
- ☺ sound, complete, and terminating: 5 kLOP

Verified Decision Procedures

for

Monadic Second-Order Logic on Strings

Functional Pearl

Dmitriy Traytel Tobias Nipkow

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