

Truly Modular (Co)datatypes

for



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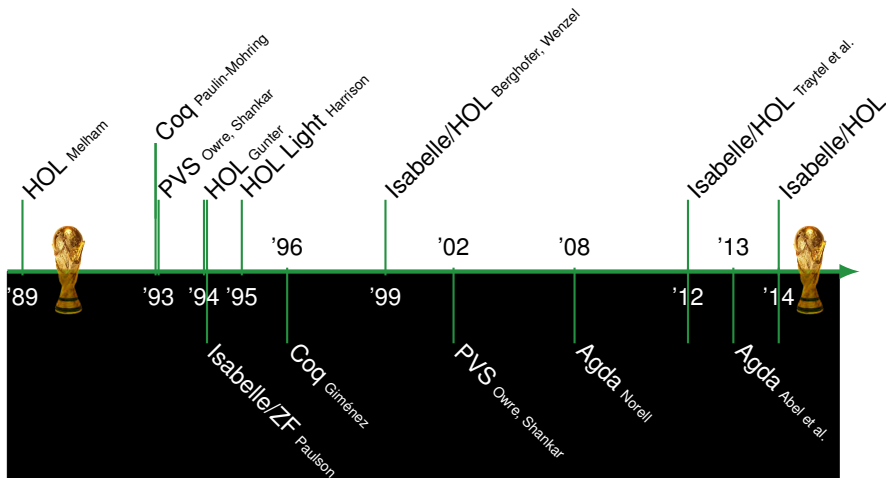


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ETH zürich

DATATYPES



CODATATYPES

Disclaimer

Isabelle/HOL was **not** harmed
by the introduction of new axioms
to achieve the following

theory Examples

imports Library

begin

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

theory Examples

imports Library

begin

datatype_new 'a list = Nil | Cons 'a "'a list"

codatatype 'a llist = LNil | LCons 'a "'a llist"

datatype_new 'a tree = Node 'a "'a tree list"

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
codatatype 'a ltree = LNode 'a "'a ltree list"
```

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
codatatype 'a ltree = LNode 'a "'a ltree list"
```

```
codatatype 'a ltreeω = LNodeω 'a "'a ltreeω llist"
```

theory Examples

imports Library

begin

datatype_new 'a list = Nil | Cons 'a "'a list"

codatatype 'a llist = LNil | LCons 'a "'a llist"

datatype_new 'a tree = Node 'a "'a tree list"

datatype_new 'a tree_ω = Node_ω 'a "'a tree_ω llist"

codatatype 'a ltree = LNode 'a "'a ltree list"

codatatype 'a ltree_ω = LNode_ω 'a "'a ltree_ω llist"

datatype_new 'a tree_{fset} = Node_{fset} 'a "'a tree_{fset} fset"

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
codatatype 'a ltree = LNode 'a "'a ltree list"
```

```
codatatype 'a ltreeω = LNodeω 'a "'a ltreeω llist"
```

```
datatype_new 'a treefset = Nodefset 'a "'a treefset fset"
```

```
codatatype 'a treecset = LNodecset 'a "'a treecset cset"
```

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
codatatype 'a ltree = LNode 'a "'a ltree list"
```

```
codatatype 'a ltreeω = LNodeω 'a "'a ltreeω llist"
```

```
datatype_new 'a treefset = Nodefset 'a "'a treefset fset"
```

```
codatatype 'a treecset = LNodecset 'a "'a treecset cset"
```

```
datatype_new 'a treeset = Nodeset 'a "'a treeset set"
```

```
theory Examples
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
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```
codatatype 'a llist = LNil | LCons 'a "'a llist"
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```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
datatype_new 'a treeω = Nodeω 'a "'a treeω llist"
```

```
codatatype 'a ltree = LNode 'a "'a ltree list"
```

```
codatatype 'a ltreeω = LNodeω 'a "'a ltreeω llist"
```

```
datatype_new 'a treefset = Nodefset 'a "'a treefset fset"
```

```
codatatype 'a treecset = LNodecset 'a "'a treecset cset"
```

```
datatype_new 'a treeset = Nodeset 'a "'a treeset set"
```

Unsupported recursive occurrence of type
" 'a local.tree_{set} " via type constructor "Set.set" in
type expression " 'a local.tree_{set} set "
Use the "bnf" command to register "Set.set" as a bounded
natural functor to allow nested (co)recursion through it

Outline

Examples

Bounded Natural Functors

(Co)datatypes

Primitive (Co)recursion

Closing Remarks

Outline

Examples

Bounded Natural Functors

(Co)datatypes

Primitive (Co)recursion

Closing Remarks

BNF

=

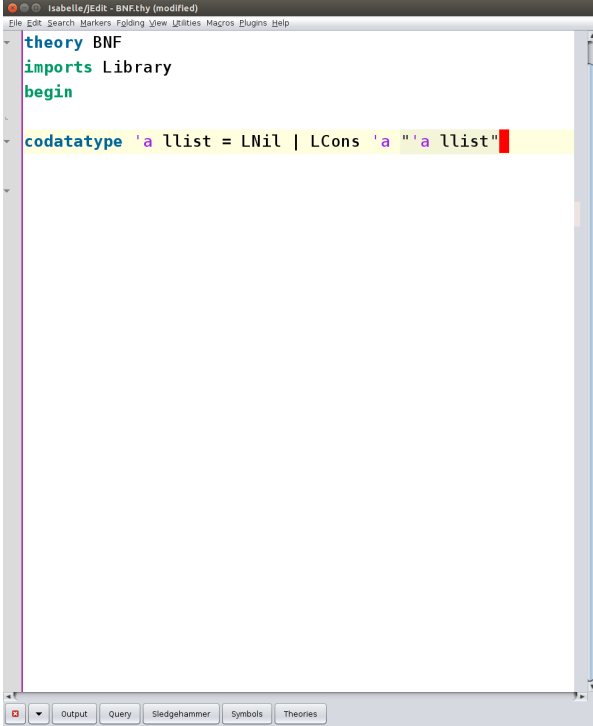
type

+

polymorphic constants
(map, set, bound, relator)

+

theorems



Isabelle/Edit - BNF.thy (modified)

File Edit Search Markers Folding View Utilities Macros Plugins Help

```
theory BNF
imports Library
begin

codatatype 'a llist = LNil | LCons 'a "'a llist"

term "map_llist :: ('a  $\Rightarrow$  'b)  $\Rightarrow$  'a llist  $\Rightarrow$  'b llist"
term "set_llist :: 'a llist  $\Rightarrow$  'a set"
term " $\mathbb{N}_0$  :: (nat  $\times$  nat) set"
term "rel_llist :: ('a  $\Rightarrow$  'b  $\Rightarrow$  bool)  $\Rightarrow$ 
      'a llist  $\Rightarrow$  'b llist  $\Rightarrow$  bool"
```

Output Query Sledgehammer Symbols Theories

theory BNF**imports** Library**begin****codatatype** 'a llist = LNil | LCons 'a "'a llist"**term** "map_llist :: ('a $\Rightarrow$ 'b) $\Rightarrow$ 'a llist $\Rightarrow$ 'b llist"**term** "set_llist :: 'a llist $\Rightarrow$ 'a set"**term** " $\mathbb{N}_0$:: (nat $\times$ nat) set"**term** "rel_llist :: ('a $\Rightarrow$ 'b $\Rightarrow$ bool) $\Rightarrow$ 'a llist $\Rightarrow$ 'b llist $\Rightarrow$ bool"**bnf** "'a llist"

map: map_llist

sets: set_llist

bd: " $\mathbb{N}_0$ "

rel: rel_llist

proof

oops

```
theory BNF
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist = LNil | LCons 'a "'a llist"
```

```
term "map_llist :: ('a  $\Rightarrow$  'b)  $\Rightarrow$  'a llist  $\Rightarrow$  'b llist"
```

```
term "set_llist :: 'a llist  $\Rightarrow$  'a set"
```

```
term " $\aleph_0$  :: (nat  $\times$  nat) set"
```

```
term "rel_llist :: ('a  $\Rightarrow$  'b  $\Rightarrow$  bool)  $\Rightarrow$   
      'a llist  $\Rightarrow$  'b llist  $\Rightarrow$  bool"
```

☒ Auto update

Search:

85%

```
proof (state): depth 0
```

```
goal (9 subgoals):
```

```
1. map_llist id = id
```

```
2.  $\bigwedge f g. \text{map\_llist } (g \circ f) = \text{map\_llist } g \circ \text{map\_llist } f$ 
```

```
3.  $\bigwedge x f g. (\bigwedge z. z \in \text{set\_llist } x \implies f z = g z) \implies$   
   map_llist f x = map_llist g x
```

```
4.  $\bigwedge f. \text{set\_llist } \circ \text{map\_llist } f = \text{op } \backslash f \circ \text{set\_llist}$ 
```

```
5. card_order  $\aleph_0$ 
```

```
6. cinfinite  $\aleph_0$ 
```

```
7.  $\bigwedge x. |\text{set\_llist } x| \leq \aleph_0$ 
```

```
8.  $\bigwedge R S. \text{rel\_llist } R \text{ OO rel\_llist } S \leq \text{rel\_llist } (R \text{ OO } S)$ 
```

```
9.  $\bigwedge R. \text{rel\_llist } R =$   
    $(\lambda x y. \exists z. \text{set\_llist } z \subseteq \{(x, y). R x y\} \wedge$   
     map_llist fst z = x  $\wedge$  map_llist snd z = y)
```

BNF

=

a semantic criterion
for legal rhs of a
(co)datatype declaration

BNF

=

a semantic criterion
for legal rhs of a
(co)datatype declaration

$_ \times _$ $_ \text{fset}$

$_ + _$ **Manually
registered
BNFs** $_ \text{mset}$

$\tau \Rightarrow _$ $_ \text{cset}$

$_ \times _$ $_ \text{fset}$
 $_ + _$ $_ \text{mset}$
 $\tau \Rightarrow _$ $_ \text{cset}$

Manually
registered
BNFs

compositions of BNFs

$\text{unit} + _ \times _$ codatatypes
 $_ \text{list}$ $_ \text{lset}$

datatypes

Automatically
derived
BNFs

$_ \times _$ $_ \text{fset}$
 $_ + _$ $_ \text{mset}$
 $\tau \Rightarrow _$ $_ \text{cset}$

Manually
registered
BNFs

$_ \Rightarrow \tau$

Non-
BNFs

$_ \text{set}$

compositions of BNFs

$\text{unit} + _ \times _$

codatatypes

$_ \text{llist}$

datatypes

$_ \text{list}$

Automatically
derived
BNFs

Outline

Examples

Bounded Natural Functors

(Co)datatypes

Primitive (Co)recursion

Closing Remarks

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
datatype_new 'a list = ctor "unit + 'a × 'a list"
```

theory Datatypes**imports** Library**begin****(*** datatype_new 'a list = Nil | Cons 'a "'a list" *)**(*** datatype_new 'a list = ctor "unit + 'a $\times$ 'a list" *)**text** {***** 'b = unit + 'a $\times$ 'b *****}

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
val (pre_bnf, ctxt) =
```

```
  bnf_of_typ @{context} @{typ "unit + 'a × 'b"};
```

```
*}
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
val (pre_bnf, ctxt) =
```

```
  bnf_of_typ @{context} @{typ "unit + 'a × 'b"};
```

```
print_term ctxt "map" (map_of_bnf pre_bnf);
```

```
print_terms ctxt "sets" (sets_of_bnf pre_bnf);
```

```
print_thms ctxt "set_map" (set_map0_of_bnf pre_bnf);
```

```
*}
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
map
```

```
λf1 f2. map_sum id (map_prod f1 f2)
```

```
sets
```

```
▪ λx. UNION (Basic_BNFs.setr x) Basic_BNFs.fsts
```

```
▪ λx. UNION (Basic_BNFs.setr x) Basic_BNFs.snds
```

```
set_map
```

```
▪ (λx. UNION (Basic_BNFs.setr x) Basic_BNFs.fsts) ◦
```

```
map_sum id (map_prod f1 f2) =
```

```
op ` f1 ◦
```

```
(λx. UNION (Basic_BNFs.setr x) Basic_BNFs.fsts)
```

```
▪ (λx. UNION (Basic_BNFs.setr x) Basic_BNFs.snds) ◦
```

```
map_sum id (map_prod f1 f2) =
```

```
op ` f2 ◦
```

```
(λx. UNION (Basic_BNFs.setr x) Basic_BNFs.snds)
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
val (pre_bnf, ctxt) =
```

```
  bnf_of_typ @{context} @{typ "unit + 'a × 'b"};
```

```
*}
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
val (pre_bnf, ctxt) =
```

```
  bnf_of_typ @{context} @{typ "unit + 'a × 'b"};
```

```
val ({bnfs = [bnf],
```

```
      ctor_co_induct = ctor_induct, ...}, ctxt') =
```

```
  lfp ctxt "list" pre_bnf;
```

```
*}
```

```
theory Datatypes
imports Library
begin

(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
text {* 'b = unit + 'a × 'b *}

ML_command {*
  val (pre_bnf, ctxt) =
    bnf_of_typ @{context} @{typ "unit + 'a × 'b"};
  val ({bnfs = [bnf],
        ctor_co_induct = ctor_induct, ...}, ctxt') =
    lfp ctxt "list" pre_bnf;

  print_term ctxt' "map" (map_of_bnf bnf);
  print_thm ctxt' "induction" ctor_induct;

  *}
}
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
(* datatype_new 'a list = Nil | Cons 'a "'a list" *)
```

```
(* datatype_new 'a list = ctor "unit + 'a × 'a list" *)
```

```
text {* 'b = unit + 'a × 'b *}
```

```
ML_command {*
```

```
map
```

```
local.list.map_list
```

```
induction
```

```
( $\wedge$ xb.
```

```
( $\wedge$ z.  $z \in \text{UNION (Basic\_BNFs.setr xb)}$ 
```

```
Basic_BNFs.snds  $\Rightarrow$ 
```

```
P z)  $\Rightarrow$ 
```

```
P (local.list.ctor_list xb))  $\Rightarrow$ 
```

```
pr
```

```
print_thm ctxt' "induction" ctor_induct;
```

```
*}
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
term "ctor_list :: unit + 'a × 'a list ⇒ 'a list"
```

```
thm Nil_def Cons_def
```

- Nil $\equiv$ ctor_list (Inl ())
- Cons $\equiv$ $\lambda x_{21}' x_{22}. \text{ctor_list } (\text{Inr } (x_{21}', x_{22}))$

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
term "ctor_list :: unit + 'a × 'a list ⇒ 'a list"
```

```
thm Nil_def Cons_def
```

```
thm list.ctor_induct list.induct
```

```
▪ (∧xb. (∧z. z ∈ set2_pre_list xb ⇒ P z) ⇒  
  P (ctor_list xb)) ⇒  
  P z  
▪ P Nil ⇒  
  (∧x1 x2. P x2 ⇒ P (Cons x1 x2)) ⇒ P list
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
term "ctor_list :: unit + 'a × 'a list ⇒ 'a list"
```

```
thm Nil_def Cons_def
```

```
thm list.ctor_induct list.induct
```

```
term "ctor_rec_list ::
```

```
  (unit + 'a × 'a list × 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"
```

```
term "rec_list ::
```

```
  'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"
```

```
thm list.rec
```

```
• rec_list f1 f2 Nil = f1  
• rec_list f1 f2 (Cons x21 x22) =  
  f2 x21 x22 (rec_list f1 f2 x22)
```

```
theory Datatypes
```

```
imports Library
```

```
begin
```

```
datatype_new 'a list = Nil | Cons 'a "'a list"
```

```
term "ctor_list :: unit + 'a × 'a list ⇒ 'a list"
```

```
thm Nil_def Cons_def
```

```
thm list.ctor_induct list.induct
```

```
term "ctor_rec_list ::
```

```
  (unit + 'a × 'a list × 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"
```

```
term "rec_list ::
```

```
  'b ⇒ ('a ⇒ 'a list ⇒ 'b ⇒ 'b) ⇒ 'a list ⇒ 'b"
```

```
thm list.rec
```

```
primrec tails :: "'a list ⇒ 'a list list" where
```

```
  "tails Nil = Cons Nil Nil"
```

```
| "tails (Cons x xs) = Cons (Cons x xs) (tails xs)"
```

```
theory Codatatypes
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
theory Codatatypes
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dctor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  'b  $\Rightarrow$  'a llist"
```

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theory Codatatypes
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  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dctor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$  (* is LNil? *)
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
    'b  $\Rightarrow$  'a llist"
```

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theory Codatatypes
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codatatype 'a llist =
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  LNil | LCons (lhd: 'a) (ltl: "'a llist")
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term "dctor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* is LNil? *)
```

```
  ('b  $\Rightarrow$  'a)  $\Rightarrow$       (* lhd *)
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  -  $\Rightarrow$ 
```

```
  'b  $\Rightarrow$  'a llist"
```

```
theory Codatatypes
```

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imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dctor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* is LNil? *)
```

```
  ('b  $\Rightarrow$  'a)  $\Rightarrow$         (* lhd *)
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* ltl: stop? *)
```

```
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$   (* ltl: end *)
```

```
  ('b  $\Rightarrow$  'b)  $\Rightarrow$        (* ltl: continue *)
```

```
  'b  $\Rightarrow$  'a llist"
```

```
theory Codatatypes
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dtor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* is LNil? *)
```

```
  ('b  $\Rightarrow$  'a)  $\Rightarrow$         (* lhd *)
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* ltl: stop? *)
```

```
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$   (* ltl: end *)
```

```
  ('b  $\Rightarrow$  'b)  $\Rightarrow$         (* ltl: continue *)
```

```
  'b  $\Rightarrow$  'a llist"
```

```
thm llist.disc_corec llist.sel_corec
```

```

▪ p a  $\Rightarrow$  corec_llist p g21 q22 g221 g222 a = LNil
▪  $\neg$  p a  $\Rightarrow$ 
  corec_llist p g21 q22 g221 g222 a  $\neq$  LNil
▪ p a  $\Rightarrow$ 
  lhd (corec_llist p g21 q22 g221 g222 a) = g21 a
▪  $\neg$  p a  $\Rightarrow$ 
  ltl (corec_llist p g21 q22 g221 g222 a) =
  (if q22 a then g221 a
   else corec_llist p g21 q22 g221 g222
    (g222 a))
```

```
theory Codatatypes
imports Library
begin

codatatype 'a llist =
  LNil | LCons (lhd: 'a) (ltl: "'a llist")

term "dtr_corec_llist ::
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
term "corec_llist ::
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* is LNil? *)
  ('b  $\Rightarrow$  'a)  $\Rightarrow$          (* lhd *)
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* ltl: stop? *)
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$   (* ltl: end *)
  ('b  $\Rightarrow$  'b)  $\Rightarrow$        (* ltl: continue *)
  'b  $\Rightarrow$  'a llist"
thm llist.disc_corec llist.sel_corec

primcorec ltails :: "'a llist  $\Rightarrow$  'a llist llist" where
```

```
theory Codatatypes
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dtor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* is LNil? *)
```

```
  ('b  $\Rightarrow$  'a)  $\Rightarrow$         (* lhd *)
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$       (* ltl: stop? *)
```

```
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$   (* ltl: end *)
```

```
  ('b  $\Rightarrow$  'b)  $\Rightarrow$         (* ltl: continue *)
```

```
  'b  $\Rightarrow$  'a llist"
```

```
thm llist.disc_corec llist.sel_corec
```

```
primcorec ltails :: "'a llist  $\Rightarrow$  'a llist llist" where
```

```
  "ltails xs  $\neq$  LNil"
```

```

theory Codatatypes
imports Library
begin

codatatype 'a llist =
  LNil | LCons (lhd: 'a) (ltl: "'a llist")

term "dctor_corec_llist ::
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"

term "corec_llist ::
  ('b  $\Rightarrow$  bool)  $\Rightarrow$  (* is LNil? *)
  ('b  $\Rightarrow$  'a)  $\Rightarrow$  (* lhd *)
  ('b  $\Rightarrow$  bool)  $\Rightarrow$  (* ltl: stop? *)
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$  (* ltl: end *)
  ('b  $\Rightarrow$  'b)  $\Rightarrow$  (* ltl: continue *)
  'b  $\Rightarrow$  'a llist"

thm llist.disc_corec llist.sel_corec

primcorec ltails :: "'a llist  $\Rightarrow$  'a llist llist" where
  "ltails xs  $\neq$  LNil"
  | "lhd (ltails xs) = xs"

```

```
theory Codatypes
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
term "dctor_corec_llist ::
```

```
  ('b  $\Rightarrow$  unit + 'a  $\times$  ('a llist + 'b))  $\Rightarrow$  'b  $\Rightarrow$  'a llist"
```

```
term "corec_llist ::
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$  (* is LNil? *)
```

```
  ('b  $\Rightarrow$  'a)  $\Rightarrow$  (* lhd *)
```

```
  ('b  $\Rightarrow$  bool)  $\Rightarrow$  (* ltl: stop? *)
```

```
  ('b  $\Rightarrow$  'a llist)  $\Rightarrow$  (* ltl: end *)
```

```
  ('b  $\Rightarrow$  'b)  $\Rightarrow$  (* ltl: continue *)
```

```
  'b  $\Rightarrow$  'a llist"
```

```
thm llist.disc_corec llist.sel_corec
```

```
primcorec ltails :: "'a llist  $\Rightarrow$  'a llist llist" where
```

```
  "ltails xs  $\neq$  LNil"
```

```
| "lhd (ltails xs) = xs"
```

```
| "ltl (ltails xs) =
```

```
  (if xs = LNil then LNil else ltails (ltl xs))"
```

Outline

Examples

Bounded Natural Functors

(Co)datatypes

Primitive (Co)recursion

Closing Remarks

Isabelle/!Edit - Primcorec.thy (modified)

File Edit Search Markers Folding View Utilities Macros Plugins Help

```
theory Primcorec
imports Library
begin

codatatype 'a llist =
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

Output Query Sledgehammer Symbols Theories

```
theory Primcorec
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
primcorec fpdest :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

Isabelle/!Edit - Primcorec.thy (modified)

File Edit Search Markers Folding View Utilities Macros Plugins Help

```
theory Primcorec
imports Library
begin

codatatype 'a llist =
  LNil | LCons (lhd: 'a) (ltl: "'a llist")

primcorec fpdest :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
  "f a = a ⇒ fpdest f a = LNil"
```

Output Query Sledgehammer Symbols Theories

Isabelle/!Edit - Primcorec.thy (modified)

File Edit Search Markers Folding View Utilities Macros Plugins Help

```
theory Primcorec
imports Library
begin

codatatype 'a llist =
  LNil | LCons (lhd: 'a) (ltl: "'a llist")

primcorec fpdest :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
  "f a = a ⇒ fpdest f a = LNil"
| "lhd (fpdest f a) = f a"
| "ltl (fpdest f a) = fpdest f (f a)"
```

Output Query Sledgehammer Symbols Theories

```
theory Primcore
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
primcorec fpdest :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

```
  "f a = a ⇒ fpdest f a = LNil"
```

```
| "lhd (fpdest f a) = f a"
```

```
| "ltl (fpdest f a) = fpdest f (f a)"
```

```
primcorec fpcons :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

```
  "f a = a ⇒ fpcons f a = LNil"
```

```
| "f a ≠ a ⇒ fpcons f a = LCons (f a) (fpcons f (f a))"
```

```
theory Primcorec
```

```
imports Library
```

```
begin
```

```
codatatype 'a llist =
```

```
  LNil | LCons (lhd: 'a) (ltl: "'a llist")
```

```
primcorec fpdest :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

```
  "f a = a ⇒ fpdest f a = LNil"
```

```
| "lhd (fpdest f a) = f a"
```

```
| "ltl (fpdest f a) = fpdest f (f a)"
```

```
primcorec fpcons :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

```
  "f a = a ⇒ fpcons f a = LNil"
```

```
| "f a ≠ a ⇒ fpcons f a = LCons (f a) (fpcons f (f a))"
```

```
primcorec fpcode :: "('a ⇒ 'a) ⇒ 'a ⇒ 'a llist" where
```

```
  "fpcode f a = (let fa = f a in
```

```
    if fa = a then LNil else LCons fa (fpcode f fa))"
```

```
theory Primrec
imports Library
begin
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
theory Primrec
```

```
imports Library
```

```
begin
```

```
datatype_new 'a tree = Node 'a "'a tree list"
```

```
primrec sum_tree :: "nat tree  $\Rightarrow$  nat" where
```

```
  "sum_tree (Node a ts) =
```

```
    a + sum_list (map sum_tree ts)"
```

```
theory Primrec
imports Library
begin

datatype_new 'a tree = Node 'a "'a tree fset"

primrec sum_tree :: "nat tree  $\Rightarrow$  nat" where
  "sum_tree (Node a ts) =
    a + sum_fset (fimage sum_tree ts)"
```

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Closing Remarks

Some Numbers

development size

19 kLoC ML, 7 kLoC thy

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In the **Coinductive** library,
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performance: mutual ($m = 15$)

$$\frac{\text{new}}{\text{old}} \approx 2.5$$

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$$\frac{\text{new}}{\text{old}} = 0.04$$

Some Numbers

development size

19 kLoC ML, 7 kLoC thy

performance: mutual (m = 15)

$$\frac{\text{new}}{\text{old}} \approx 2.5$$

automation gains

In the **Coinductive** library,
we now prove 36% more
lemmas in 11% fewer lines

performance: nested (n = 15)

$$\frac{\text{new}}{\text{old}} = 0.04$$

performance: real world

The **IsaFoR** session Proof-
Checker now compiles in 10
minutes instead of 50!

René Thiemann

Related Talks

Lochbihler, Hölzl (next talk)

Recursive Functions on Lazy Lists via Domains and Topologies

Blanchette, Popescu, Traytel (tomorrow 12:15)

Cardinals in Isabelle/HOL

Blanchette, Popescu, Traytel (Saturday 11:45, IJCAR)

Unified Classical Logic Completeness: A Coinductive Pearl

Truly Modular (Co)datatypes



Jasmin Blanchette
Lorenz Panny
Dmitriy Traytel

Johannes Hölzl
Andrei Popescu

Technische Universität München

Andreas Lochbihler
