Admissible Types-to-PERs Relativization

in

Higher-Order Logic

Andrei Popescu

University of Sheffield

Dmitriy Traytel

University of Copenhagen

Proof Assistants, a.k.a. Interactive Theorem Provers

ACL2 Agda Coq HOL4 HOL Light HOL-ω

Isabelle/HOL Lean Mizar Nuprl Matita PVS

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Logical foundations: a form of Higher-Order Logic (HOL) or Dependent Type Theory

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Verification Success Stories

Making the news: seL4, CompCert, CakeML,
Kepler's, Four Color, Odd Order, Gödel's theorems

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Libraries of mathematical results
Automated proof methods
Abstraction/modularization mechanisms
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Two Little Pigs

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group : $(\alpha \Rightarrow \alpha \Rightarrow \alpha) \Rightarrow \alpha \Rightarrow \text{bool}$

Two Little Pigs

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$$
\begin{aligned}
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&(\forall x_\alpha, y_\alpha, z_\alpha.\ (x * y) * z = x * (y * z)) \wedge \\
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proved Abel-Galois-Klein-Lie-Ruffini
in 10 kLoI and 2 person months

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to use AGKLR for symmetric groups:

1. $\alpha\ \text{perm} = \{f :: \alpha \Rightarrow \alpha.\ \text{bij}\ f\}$
2. lift results about bijections to α perm
   (10 kLoI and 2 person months)

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in 30 kLoI and 6 person months

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Type-based vs. Set-based

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Our Contribution

A logically safe mechanism for reducing the bureaucracy of formal developments in HOL-based proof assistants

HOL in One Line

Rank-1 Polymorphic Simply-Typed λ-Calculus with Hilbert Choice and Infinity

HOL in One Slide

Types

$\tau = \alpha \mid (\tau, \ldots, \tau)\ \kappa$ *bool* *ind* $\Rightarrow$

HOL in One Slide

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Terms

$t = x_\tau \mid c_\tau \mid t\ t \mid \lambda x_\tau.\ t$ $=_{\alpha \Rightarrow \alpha \Rightarrow bool}$ $\text{choice}_{(\alpha \Rightarrow bool) \Rightarrow \alpha}$

HOL in One Slide

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Definitions

Constant $c_\tau \equiv t$

Type $t_{\sigma\Rightarrow bool}\ x \longrightarrow \tau \equiv^{\text{Rep}_{\tau\Rightarrow\sigma}}_{\text{Abs}_{\sigma\Rightarrow\tau}}\ t$

$\text{True}_{bool} \equiv (\lambda x_{bool}.\ x) = (\lambda x.\ x)$

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Axioms

refl $x_\alpha = x$

sub $x_\alpha = y \longrightarrow p\ x \longrightarrow p\ y$

inf $\exists z, s.\ (\forall x.\ \neg\ (s\ x = z)) \wedge (\forall x, y.\ s\ x = s\ y \longrightarrow x = y)$

ch $p_{\alpha \Rightarrow bool}\ x \longrightarrow p\ (\text{choice}\ p)$

HOL in One Slide

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Deduction

$$\frac{\varphi \in D \cup \Gamma}{D; \Gamma \vdash \varphi} \qquad \frac{D; \Gamma \vdash \varphi \qquad \alpha \notin \Gamma}{D; \Gamma \vdash \varphi[\sigma/\alpha]} \qquad \frac{D; \Gamma \vdash \varphi \qquad x_\sigma \notin \Gamma}{D; \Gamma \vdash \varphi[t/x_\sigma]} \qquad \frac{}{D; \Gamma \vdash (\lambda x.\ t)\ s = t[s/x_\sigma]}$$

$$\frac{D; \Gamma \vdash \varphi \longrightarrow \psi \qquad D; \Gamma \vdash \varphi}{D; \Gamma \vdash \psi} \qquad \frac{D; \Gamma \cup \{\varphi\} \vdash \psi}{D; \Gamma \vdash \varphi \longrightarrow \psi} \qquad \frac{D; \Gamma \vdash f\, x_\sigma = g\, x_\sigma \qquad x_\sigma \notin \Gamma}{D; \Gamma \vdash f = g}$$

HOL in Practice

And that's it! No induction, no datatypes, no recursion in the kernel!

These high-level mechanisms are defined from the (non-recursive) HOL primitives.

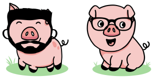
↓

Fewer opportunities to have soundness bugs in the kernel.

Back to  

Type-based formulation φ and its relativized set-based counterpart φ^{rlt}:

φ	$\forall e_\alpha.\ \forall *_{\alpha\Rightarrow\alpha\Rightarrow\alpha}.\ \text{group}\ (*)\ e \longrightarrow (\forall x_\alpha, y_\alpha, z_\alpha.\ x * y = x * z \longrightarrow y = z)$
φ^{rlt}	$\forall e_\alpha.\ *_{\alpha\Rightarrow\alpha\Rightarrow\alpha}.\ \forall A_{\alpha\ \text{set}}.\ \text{closed}\ A\ (*)\ e \longrightarrow$ $\text{group}^{\text{rlt}}\ A\ (*)\ e \longrightarrow (\forall x_\alpha, y_\alpha, z_\alpha{\in}A.\ x * y = x * z \longrightarrow y = z)$

Back to 

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Our goal: Formally (and automatically) infer φ^{rlt} from φ.

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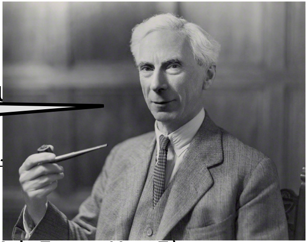

Type- … sed cou …

> The method of "postulating" what we want has many advantages; they are the same as the advantages of theft over honest toil.

$z_\alpha .\ x\ *$ …

φ^{rlt}

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But first: It was not clear how φ^{rlt} looks like in general.
What *structure* and *properties* do we need for relativization?

Towards a General Definition of Types-To-Sets

Example: the second-order property of left-inverse function uniqueness

$$\varphi \quad \begin{array}{|l} \forall e_\alpha.\ \forall *_{\alpha \Rightarrow \alpha \Rightarrow \alpha}.\ \text{group}\ (*)\ e \longrightarrow \\ (\forall inv_{\alpha \Rightarrow \alpha}, inv'_{\alpha \Rightarrow \alpha}.\ (\forall x_\alpha.\ inv\ x * x = e) \wedge (\forall x_\alpha.\ inv'\ x * x = e) \longrightarrow inv = inv') \\ \\ \end{array}$$

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Relaxed equalities + restricted domains = PERs (partial equivalence relations)

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We generalize from Types-To-Sets to Types-To-PERs

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Another example: iterated multiplication distributes over list append

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Towards a General Definition of Types-To-Sets

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Want that relationally parameteric functions are not affected by relativization.
Why? PER-relativization turns a term into its ”closest” PER-parametric counterpart.

Summary of the Desiderata for Relativization

Term relativization RLT and relational type interpretation RIN, such that:

1. RIN should be compatible with the container-type relators
2. RLT should generalize FOL-relativization
3. RLT(t) should recover t when instantiating relations to equalities
4. RLT(t) should be PER-parametric
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6. RLT should preserve provability (i.e., be admissible)

Defining RIN : Type ⟶ Term **and** RLT : Term ⟶ Term

$\mathsf{RIN}(\alpha) = R_{\alpha\ \mathsf{rel}}$ $\qquad \mathsf{RIN}(\mathsf{bool}) = =_{\mathsf{bool\ rel}}$ $\qquad \mathsf{RIN}(\sigma \Rightarrow \tau) = \mathsf{RIN}(\sigma) \Rightarrow \mathsf{RIN}(\tau)$

$$\mathsf{RIN}(\tau) = \mathsf{choice}\ (\lambda P_{\tau\ \mathsf{rel}}.\ \exists f_{\tau\Rightarrow\sigma}.\ \mathsf{bijUpto}\ f\ P\ \mathsf{RIN}(\sigma)_{\restriction \mathsf{RLT}(t)})$$

if $\tau \equiv t$ is a type definition where $t : \sigma \Rightarrow \mathsf{bool}$ and τ has the form $(\sigma_1, \ldots, \sigma_m)\kappa$

$\mathsf{RLT}(x_\sigma) = x_\sigma$ $\qquad \mathsf{RLT}(t_1\ t_2) = \mathsf{RLT}(t_1)\ \mathsf{RLT}(t_2)$ $\qquad \mathsf{RLT}(\lambda x_\sigma.\ t) = \lambda x_\sigma.\ \mathsf{RLT}(t)$

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Equality mapped to the PER-relational interpretation

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The (notoriously non-parametric) Hilbert Choice needs special treatment

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– Only behaves well when defining predicates t are "wide" enough: $|t| \geq |\mathsf{RLT}(t)/\mathsf{RIN}(\sigma)|$.

– Choice of P needs flexibility to correctly capture container types.

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Desiderata:

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The Result's Scope

Empirical Conjecture: Type definitions of interest in HOL developments are wide.

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Evidence:

- type definitions involved in defining (co)datatypes (which are already 99% of the cases of interest) are wide
- and so are the non-(co)datatypes from the Isabelle/HOL distribution

Relativization Infrastructure

```
definition group where
  "group tms e ⟷
     (∀x y z. tms (tms x y) z = tms x (tms y z)) ∧
     (∀x. tms x e = x ∧ tms e x = x) ∧
     (∀x. ∃y. tms x y = e ∧ tms y x = e)"

lemma lemma3_aux1:"group tms e ⟶ foldl tms x ys = tms x (foldl tms e ys)" [5 lines]
lemma lemma3_aux2: "group tms e ⟶ foldl tms x (xs @ ys) = tms (foldl tms x xs) (foldl tms e ys)" [1 lines]
lemma lemma3: "group tms e ⟶ (∀xs ys. foldl tms e (xs @ ys) = tms (foldl tms e xs) (foldl tms e ys))" [1 lines]

local_setup ‹RLCST @{term group}›

lemma group_rlt_alt:
  "neper R ⟹ group_rlt R tms e ⟷
    (∀x y z. R x x ∧ R y y ∧ R z z ⟶ R (tms (tms x y) z) (tms x (tms y z))) ∧
    (∀x. R x x ⟶ R (tms x e) x ∧ R (tms e x) x) ∧
    (∀x. R x x ⟶ (∃y. R y y ∧ R (tms x y) e ∧ R (tms y x) e))"
  unfolding group_rlt_def by auto

local_setup ‹RLTHM @{binding lemma3_rlt} @{thm lemma3}›

lemma lemma3_rlt_readable:
  assumes "neper R" "R e e" "(R ===> R ===> R) tms tms"
  shows "group_rlt R tms e ⟶
  (∀xs ys. rel_list R xs xs ⟶ rel_list R ys ys ⟶ R (foldl tms e (xs @ ys)) (tms (foldl tms e xs) (foldl tms e ys)))"
  supply assms(1)[simp] assms(1)[THEN list.rrel_neper, simp] using lemma3_rlt[OF assms, simplified]
  by (auto simp only: rrel_alt rlt_param neper assms(1))
```

Wideness Proofs

```
typedef (overloaded) 'a poly = "{f :: nat ⇒ 'a::zero. ∀∞ n. f n = 0}"
  morphisms coeff Abs_poly
  by (auto intro!: ALL_MOST)
```

200 lines later

```
wide_typedef poly rel: rel_poly rep: "λR1 R2. gg R1 R2 ∘ coeff"
  subgoal using rel_poly_neper .
  subgoal using rel_poly_eq .
  subgoal using bij_upto_transportFromRaw[OF poly.type_definition_poly rel_poly_def,
     unfolded isPoly_def[abs_def, symmetric], OF bij_upto_gg]
  unfolding isPoly_rlt_def .
  subgoal using gg_eq by simp .
```

Wideness Proofs

$\alpha\ \mathit{fset}^*$	finite sets over α
$\alpha\ \mathit{cset}^*$	countable sets over α
$\alpha\ \mathit{multiset}^*$	multisets (bags) over α
$(\alpha, \beta)\ \mathit{fmap}^*$	partial functions of finite support between α and β
$\alpha\ \mathit{biject}$	bijections on α
$\alpha\ \mathit{dlist}^*$	non-repetitive lists over α
$(\alpha, \beta)\ \mathit{alist}^*$	association lists with values in α and keys in β
$(\alpha, \beta)\ \mathit{node}$	the pre-datatype universe by Berghofer and Wenzel [1999]

$\alpha\ \mathit{filter}^*$	filters on the powerset of α
$\alpha\ \mathit{poly}$	polynomials with α-coefficients
$\alpha\ \mathit{fls}$	formal Laurent series with α-coefficients
$(\alpha, \beta)\ \mathit{comm}^*$	commutative α-operators with values in β
$\alpha\ \mathit{comparator}$	comparison functions on α
$\alpha\ \mathit{bit0},\ \alpha\ \mathit{bit1}$	finite types used for binary representation

,

Related Work

Relational interpretation

- Reynolds 1983. Types, abstraction, and parametric polymorphism
- Wadler 1989. Theorems for free!
- Bernardy et al. 2012. Extension to DTT

Previous work on Types-To-Sets in Isabelle

- Kunčar & Popescu 2015. Local Typedef axiom
- Immler & Zhan 2019. Divasón et al. 2020. Large case studies
- Milehins 2022. Types-To-Sets conversion tool

PER constructions in DTT

- Constable et al. 1986. Subset types and quotient types in Nuprl
- Barthe et al. 2003. (Partial) setoids in Agda and Coq
- Altenkirch et al. 2019. Types to setoids in Martin-Löf type theory

Take Home Message

Motto: Prove easily (type-based) and still be flexible (PER-based)!

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And don't worry:
Very likely (if only wide types are involved), you are not reasoning outside of HOL!

Admissible Types-to-PERs Relativization

in

Higher-Order Logic

Andrei Popescu

University of Sheffield

Dmitriy Traytel

University of Copenhagen